Endomorphisms of Banach algebras of infinitely differentiable functions on compact plane sets

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February 1, 2008

This note is a sequel to [7] where we investigated the endomorphisms of a certain class of Banach algebras of infinitely differentiable functions on the unit interval.

Start with a perfect, compact plane set X. We say that a complex-valued function f defined on X is *complex-differentiable* at a point $a \in X$ if the limit

$$f'(a) = \lim_{z \to a,\ z \in X} \frac{f(z) - f(a)}{z - a}$$

exists. We call $f'(a)$ the *complex derivative* of f at a. Using this concept of derivative, we define the terms *complex–differentiable on* X, *continuously complex–differentiable on* X, and *infinitely complex–differentiable on* X in the obvious way. We denote the n-th complex derivative of f at a by $f^{(n)}(a)$, and we denote the set of infinitely differentiable functions on X by $D^\infty(X)$.

Let M_n be a sequence of positive numbers satisfying $M_0 = 1$ and $\begin{pmatrix} m+n \\ n \end{pmatrix} \leq \frac{M_{m+n}}{M_m M_n}$, and let $D(X, M) = \{f \in D^\infty(X) : \|f\| = \sum_{n=0}^{\infty} \frac{\|f^{(n)}\|_\infty}{M_n} < \infty\}$. With pointwise addition and multiplication, $D(X, M)$ is a commutative normed algebra which is not necessarily complete.

Clearly all polynomials when restricted to X belong to each $D(X, M)$. It was further proved in [2] that the algebra $D(X, M)$ includes all the rational functions with poles off X if and only if

$$\lim_{n \to \infty} \left(\frac{n!}{M_n} \right)^{\frac{1}{n}} = 0. \tag{1}$$

We say that M_n is a *nonanalytic sequence* if (1) holds [2].

These algebras were considered by Dales and Davie in connection with several then open questions. In [2] examples of such Banach algebras were constructed; one gave an example of a commutative semisimple Banach algebra for which the peak points were of first category in the Silov boundary, and a second was an example of a commutative semisimiple Banach algebra B and a discontinuous function F acting on B.

From a different direction, we were led to these algebras in connection with the study of compact endomorphisms of commutative semisimple Banach algebras. On the basis of many early examples, it was conjectured that all nonzero compact endomorphisms of regular commutative semisimple Banach algebras with connected maximal ideal spaces were essentially point evaluations. Our first counterexample was the endomorphism $T : Tf(x) = f(\frac{x}{2})$ on $D([0,1], n!^2)$. [6]

This led to the question of determining all endomorphisms of $D([0,1], n!^2)$ and more generally determining the endomorphisms of other algebras $D(X, M)$. In [7], the question was settled for $X = [0,1]$ in many cases when the weights M_n were nonanalytic.

In this paper we look at more general perfect, compact subsets of the plane, and investigate the extent to which the results for the interval extend to this setting. In particular we shall give a variety of results in the case of the closed unit disk. We shall also partially resolve some of the problems left open for the interval in [7].

In general, the normed algebra $D(X, M)$ need not be complete. However, if the compact set X is a finite union of uniformly regular sets, [1] then $D(X, M)$ is complete for every weight M_n. Such sets include $[0,1]$ and $\bar{\Delta}$ where Δ is the open unit disc.

We recall further results from [2]. Suppose $D_R(X, M)$ is the closed subalgebra of $D(X, M)$ generated by the rational functions with poles off X. If M_n is nonanalytic, and X is uniformly regular, then $D_R(X, M)$ is *natural* meaning that the maximal ideal space of $D_R(X, M)$ is X. Further, for nicely shaped X, it was shown in [4] that $D_R(X, M) = D(X, M)$. More can be said in the cases of the unit disc, Δ, and unit interval. Indeed, it was shown in [3] that the polynomials are dense in $D(\bar{\Delta}, M)$ and in [9] that the polynomials are dense in $D([0,1], M)$.

In contrast, when the weight $M_n = n!$, the maximal ideal space of the

[1] A compact plane set X is *uniformly regular* if, for all $z, w \in X$, there is a rectifiable arc in X joining z to w, and the metric given by the geodesic distance between points of X is uniformly equivalent to the Euclidean metric. [2]

Banach algebra $D([0,1],n!)$ equals $\{\lambda : \text{dist}(\lambda,[0,1]) \le 1\}$. If $\sum_{n=0}^{\infty} \frac{M_n}{M_{n+1}} = \infty$ then the algebra is *quasi-analytic* in the sense that if $f^{(n)}(a) = 0$ for all n, then $f = 0$.

Some examples when $X = [0,1]$: If $M_n = n!^{\alpha}$, $\alpha > 1$, the algebra $D([0,1],M)$ is natural and regular, while if $M_n = n![\log(n+1)]^n$ the algebra is natural and quasi-analytic. If $M_n = n!$ then the algebra is quasi-analytic and not natural.

From general principles, if T is a nonzero endomorphism of a commutative semisimple Banach algebra B with maximal ideal space Φ_B, then there exists a continuous map $\phi : \Phi_B \to \Phi_B$ such that $\widehat{Tf}(x) = \hat{f}(\phi(x))$ for all $f \in B$, $x \in \Phi_B$. For a natural Banach algebra $D(X,M)$, our question can be restated as determining conditions on $\phi : X \to X$ for which $f \circ \phi \in D(X,M)$ for all $f \in D(X,M)$.

The results in Parts A and B concern two types of theorems. One will deal with arbitrary nonanalytic weights M_n and $\phi : X \to X$ which satisfy an additional analyticity property. Then the analyticity property on ϕ will be removed and weights M_n constructed such that the theorems hold. The paper will then conclude in Part C with a detailed study when $X = \bar{\Delta}$.

Finally, in the remainder of this note any unlabeled norm $\|\cdot\|$ will denote the sup norm of the function on X. The symbol $\|\cdot\|_{D(X,M)}$ will be used for the algebra norm.

Part A

The following was proved in [7] for $X = [0,1]$, but an examination of the proof shows that the reasoning does not depend on $[0,1]$. Since references will be made to this proof, for the convenience of the reader we reproduce the proof of part (a) from [7] in the more general form. We remark, too, that the proof of part (b) also goes through with minor modifications.

Theorem 1: Let X be a perfect, compact plane set and M_n be a nonanalytic weight sequence.

(a) Suppose $\phi \in D^{\infty}(X)$, $\phi : X \to X$, $\limsup_{k\to\infty} (\frac{\|\phi^{(k)}\|}{k!})^{1/k}$ is finite and $\|\phi'\|_{\infty} < 1$. Then ϕ induces an endomorphism of $D(X,M)$.

(b) If, in addition, there exists $B > 0$ such that $\frac{M_m}{m!}\frac{n!}{M_n} \le \frac{B}{m^{n-m}}$ for $n \ge m \ge 1$, and $\{\frac{\|\phi^{(k)}\|}{k!}\}$ is bounded, then $\|\phi'\|_{\infty} \le 1$ is sufficient for ϕ to induce an endomorphism of $D(X,M)$.

Proof:

(a) Suppose ϕ satisfies the hypotheses. Let $F \in D(X, M)$. We show that $F \circ \phi \in D(X, M)$. The following equality for higher derivatives of composite functions is known as Faà di Bruno's formula.

$$\frac{d^n}{dx^n}(F \circ \phi) = \sum_{m=0}^{n} F^{(m)}(\phi) \mathbf{\Sigma} \frac{n!}{a_1! a_2! \cdots a_n!} \frac{(\phi')^{a_1} (\phi'')^{a_2} \cdots (\phi^{(n)})^{a_n}}{1!^{a_1} 2!^{a_2} \cdots n!^{a_n}}$$

where the inner sum Σ is over non-negative integers $a_1, a_2, \cdots, a_n$ satisfying $a_1 + a_2 + \cdots + a_n = m$ and $a_1 + 2a_2 + \cdots + na_n = n$.

Throughout the proof of Theorem 1 the inner sum Σ will always be over non-negative integers $a_1, a_2, \cdots, a_n$ satisfying $a_1 + a_2 + \cdots + a_n = m$ and $a_1 + 2a_2 + \cdots + na_n = n$.

Thus, Faà di Bruno's formula implies that

$$\|\frac{d^n}{dx^n}(F \circ \phi)\| \leq \sum_{m=0}^{n} \|F^{(m)}(\phi)\| \mathbf{\Sigma} \frac{n!}{a_1! a_2! \cdots a_n!} \frac{(\|\phi'\|)^{a_1} (\|\phi''\|)^{a_2} \cdots (\|\phi^{(n)}\|)^{a_n}}{1!^{a_1} 2!^{a_2} \cdots n!^{a_n}}$$

and so

$$\sum_{n=0}^{\infty} \frac{1}{M_n} \|\frac{d^n}{dx^n}(F \circ \phi)\| \leq \sum_{n=0}^{\infty} \frac{1}{M_n} \sum_{m=0}^{n} \|F^{(m)}\| \mathbf{\Sigma} \frac{n!}{a_1! a_2! \cdots a_n!} \frac{(\|\phi'\|)^{a_1} (\|\phi''\|)^{a_2} \cdots (\|\phi^{(n)}\|)^{a_n}}{1!^{a_1} 2!^{a_2} \cdots n!^{a_n}}.$$

Then, after interchanging the order of summation, we have

$$\sum_{n=0}^{\infty} \frac{1}{M_n} \|\frac{d^n}{dx^n}(F \circ \phi)\| \leq \sum_{m=0}^{\infty} \|F^{(m)}\| \sum_{n=m}^{\infty} \frac{1}{M_n} \mathbf{\Sigma} \frac{n!}{a_1! a_2! \cdots a_n!} \frac{(\|\phi'\|)^{a_1} (\|\phi''\|)^{a_2} \cdots (\|\phi^{(n)}\|)^{a_n}}{1!^{a_1} 2!^{a_2} \cdots n!^{a_n}},$$

whence

(*)

$$\sum_{n=0}^{\infty} \frac{1}{M_n} \|\frac{d^n}{dx^n}(F \circ \phi)\| \leq \sum_{m=0}^{\infty} \frac{\|F^{(m)}\|}{M_m} m! \sum_{n=m}^{\infty} \frac{1}{n!} \mathbf{\Sigma} \frac{M_m}{m!} \frac{n!}{M_n} \frac{n!}{a_1! a_2! \cdots a_n!} \frac{(\|\phi'\|)^{a_1} (\|\phi''\|)^{a_2} \cdots (\|\phi^{(n)}\|)^{a_n}}{1!^{a_1} 2!^{a_2} \cdots n!^{a_n}}.$$

Choose $\epsilon > 0$ such that $q := \|\phi'\|(1 + \sum_{k=2}^{\infty} \frac{\|\phi^{(k)}\|}{\|\phi'\| k!} \epsilon^{k-1}) < 1$. Such ϵ exists since $h(\lambda) = \sum_{k=2}^{\infty} \frac{\|\phi^{(k)}\|}{k!} \lambda^{k-1}$ is analytic near 0 and $h(0) = 0$.

Since $\lim_{k \to \infty} (\frac{k!}{M_k})^{1/k} = 0$, there exists $B > 0$ such that $\frac{k!}{M_k} < B\epsilon^k$ for all k, and since $\frac{M_m}{m!} \frac{n!}{M_n} \leq \frac{(n-m)!}{M_{n-m}}$ for $n \geq m$, we have that $(\frac{M_m}{m!})(\frac{n!}{M_n}) < B\epsilon^{n-m}$, $n \geq m$.

Therefore the inequality (*) implies that

$$\sum_{n=0}^{\infty}\frac{1}{M_n}\|\frac{d^n}{dx^n}(F\circ\phi)\| \leq \sum_{m=0}^{\infty}\frac{\|F^{(m)}\|}{M_m}m!\sum_{n=m}^{\infty}\frac{1}{n!}\mathbf{\Sigma}B\epsilon^{n-m}\frac{n!}{a_1!a_2!\cdots a_n!}\frac{\|\phi'\|^{a_1}\|\phi''\|^{a_2}\cdots\|\phi^{(n)}\|^{a_n}}{1!^{a_1}2!^{a_2}\cdots n!^{a_n}}$$

$$= B\sum_{m=0}^{\infty}\frac{\|F^{(m)}\|}{M_m}m!\sum_{n=m}^{\infty}\frac{\epsilon^n}{n!}\mathbf{\Sigma}\frac{n!}{a_1!a_2!\cdots a_n!}\frac{(\frac{\|\phi'\|}{\epsilon})^{a_1}(\frac{\|\phi''\|}{\epsilon})^{a_2}\cdots(\frac{\|\phi^{(n)}\|}{\epsilon})^{a_n}}{1!^{a_1}2!^{a_2}\cdots n!^{a_n}},$$

since $a_1 + a_2 + \cdots + a_n = m$.

It follows from [1], page 823, B, formula 3, that

$$m!\sum_{n=m}^{\infty}\frac{\epsilon^n}{n!}\mathbf{\Sigma}\frac{n!}{a_1!a_2!\cdots a_n!}\frac{(\frac{\|\phi'\|}{\epsilon})^{a_1}(\frac{\|\phi''\|}{\epsilon})^{a_2}\cdots(\frac{\|\phi^{(n)}\|}{\epsilon})^{a_n}}{1!^{a_1}2!^{a_2}\cdots n!^{a_n}} = (\sum_{k=1}^{\infty}\frac{\|\phi^{(k)}\|}{\epsilon k!}\epsilon^k)^m.$$

Moreover, from the definition of q,

$$(\sum_{k=1}^{\infty}\frac{\|\phi^{(k)}\|}{\epsilon k!}\epsilon^k)^m = \|\phi'\|^m(1+\sum_{k=2}^{\infty}\frac{\|\phi^{(k)}\|}{k!\|\phi'\|}\epsilon^{k-1})^m = q^m.$$

Therefore, since $F \in D(X, M)$ and $0 < q < 1$,

$$\sum_{n=0}^{\infty}\frac{1}{M_n}\|\frac{d^n}{dx^n}(F\circ\phi)\| \leq B\sum_{m=0}^{\infty}\frac{\|F^{(m)}\|}{M_m}q^m < \infty,$$

as required.

The weights $M_n = n!^\alpha$, $\alpha \geq 2$ and $M_n = n!n^{n^2}$ are nonanalytic weights for which the condition in (b), $\frac{M_m}{m!}\frac{n!}{M_n} \leq \frac{B}{m^{n-m}}$, holds. The condition fails for $M_n = n!^\alpha$, $1 < \alpha < 2$, and for $M_n = n!(\log(n+1))^n$.

Throughout this paper, a self-map ϕ of X will be said to be *analytic on* X if ϕ extends to an analytic function on a neighborhood of X.

The next theorem is an extension of Theorem 1(a) which is useful when X has nonempty interior.

Theorem 2 Let X be a perfect, compact plane set. Let ϕ be an analytic self-map of X. Let M_n be a nonanalytic sequence. Set $K = \{z \in X : |\phi'(z)| \geq 1\}$, and suppose that $\phi(K) \subset \text{int}(X)$. Then ϕ induces an endomorphism of $D(X, M)$.

Proof:

Choose $\epsilon > 0$ such that $\phi(\{z \in X : |\phi'(z)| \geq 1-\epsilon\}) \subset \text{int}(X)$. Set $K_1 = \{z \in X : |\phi'(z)| \leq 1-\epsilon\}$ and $K_2 = \{z \in X : |\phi'(z)| \geq 1-\epsilon\}$. Then

$X = K_1 \cup K_2$. For $f \in D(X, M)$, calculations similar to those in the proof of Theorem 1(a) show that the restriction of $f \circ \phi$ to K_1 is in $D(K_1, M)$. Further $f \circ \phi$ is analytic on a neighborhood of K_2, whence the restriction of $f \circ \phi$ to K_2 is in $D(K_2, M)$. It follows that $f \circ \phi \in D(X, M)$, as required.

We note that the algebras in these theorems were not necessarily complete.

This result may be improved if the stronger condition in part (b) is placed on the sequence M_n. That is, suppose that ϕ is an analytic self-map of X and M_n satisfies the condition in part (b) of Theorem 1. Then if $X = K_1 \cup K_2$ with K_1, K_2 compact such that $\phi(K_2) \subseteq \mathrm{int}(X)$ and $\|\phi'\|_{K_2} \leq 1$, then the Faà di Bruno calculations used in the proof of Theorem 1(b) in [7], combined with the argument above shows again that ϕ induces an endomorphism of $D(X, M)$.

However, we shall see that even in the case where $X = \bar{\Delta}$, there are analytic self-maps ϕ on $\bar{\Delta}$ which have $|\phi'(z)| \leq 1$ for all those z such that $|\phi(z)| = 1$, but for which no such decomposition into K_1 and K_2 is possible.

In a different direction, if the infinitely differentiable self-maps ϕ are not as well behaved as in the previous hypothesis, the following result shows that maps ϕ still induce endomorphisms of $D(X, M)$ provided that the sequence M_n grows rapidly enough.

Theorem 3: Let X be a perfect, compact plane set whose boundary is given by a finite union Γ of piecewise smooth Jordan curves, and such that Γ has winding number 1 about each point of $X \backslash \Gamma$ and 0 about each point of the complement of X. Let $\phi \in D^\infty(X)$ with $\phi : X \to X$. Suppose that for all $z \in \phi^{-1}(\Gamma)$ we have $|\phi'(z)| \leq 1$. Then there exists a nonanalytic algebra sequence M_n such that ϕ induces an endomorphism of $D(X, M)$.

Proof:

We choose the sequence M_n inductively. We start with $M_0 = M_1 = 1$. For $n \geq 2$, and having chosen $M_0, M_1, \ldots, M_{n-1}$, it follows from Faà di Bruno's formula that there are constants $C_{n,\phi,m} > 0$ such that for all infinitely differentiable functions F on X and all $z \in X$ we have

$$|(F \circ \phi)^{(n)}(z)| \leq \|F^{(n)}\|_\infty |\phi'(z)|^n + \sum_{m=0}^{n-1} C_{n,\phi,m} \|F^{(m)}\|_\infty.$$

Choose a relatively open set $U = U_{n,\phi} \supseteq \phi^{-1}(\Gamma)$ on which $|\phi'(z)|^n \leq 2$. Set $K = K_{n,\phi} = X \backslash U$. Then $\phi(K)$ is a compact subset of the interior of X.

Assuming that $K_{n,\phi} \neq \emptyset$, set

$$A_{n,\phi} = \frac{L}{2\pi} \sup\{|\frac{\mathrm{d}^n}{\mathrm{d}z^n}(\omega - \phi(z))^{-1}| : \omega \in \Gamma, z \in K_{n,\phi}\},$$

where L is the total length of Γ.

We see that, for $z \in K$ and F as above,

$$|(F \circ \phi)^{(n)}(z)| = \left| \frac{1}{2\pi i} \int_\Gamma F(\omega) \frac{\mathrm{d}^n}{\mathrm{d}z^n}(\omega - \phi(z))^{-1} \mathrm{d}\omega \right| \leq \|F\|_\infty A_{n,\phi}.$$

Now choose M_n large enough such that the following conditions are all satisfied: (i) $M_n \geq (n!)^2$; (ii) $\frac{M_n}{M_k M_{n-k}} \geq \binom{n}{k}$ for $0 \leq k \leq n-1$; (iii) $\sum_{m=0}^{n-1} C_{n,\phi,m} M_m / M_n \leq 2^{-n}$; (iv) $A_{n,\phi}/M_n \leq 2^{-n}$.

In the case where $K_{n,\phi} = \emptyset$, choose M_n satisfying (i) to (iii) above instead.

Having chosen the sequence M_n as above, we see that M_n is clearly a nonanalytic sequence. For $F \in D(X, M)$ we have, for all $k \geq 0$,

$$\|F^{(k)}\|_\infty \leq M_k \|F\|_{D(X,M)}.$$

Thus, by our choice of M_n, and considering separately the points z in $U_{n,\phi}$ and $K_{n,\phi}$, we have, for $n \geq 2$,

$$\frac{\|(F \circ \phi)^{(n)}\|_\infty}{M_n} \leq \max\{2^{-n}\|F\|_\infty, \frac{2\|F^{(n)}\|_\infty}{M_n} + 2^{-n}\|F\|_{D(X,M)}\}$$

$$= \frac{2\|F^{(n)}\|_\infty}{M_n} + 2^{-n}\|F\|_{D(X,M)}.$$

Thus $F \circ \phi$ is also in $D(X, M)$, as required.

An easy modification of this argument allows one sequence M_n to work for any given countable collection of such functions ϕ simultaneously. However these sequences M_n will grow very rapidly. The same argument can be used in an easier form to show that for every self-map $\phi \in D^\infty([0,1])$ with $\|\phi'\|_\infty \leq 1$ there is a nonanalytic sequence M_n, depending on ϕ, such that ϕ induces an endomorphism of $D([0,1], M)$. Similarly we can show that if the self-map $\phi \in D^\infty([0,1])$ with $\|\phi'\|_\infty > 1$ there is a nonanalytic sequence M, depending on ϕ, such that ϕ does not induce an endomorphism of $D([0,1], M)$. This will follow from the more general result, Theorem 5.

We now make a further definition. If X is a subset of $\mathbf{C}$ and if $c \in X$, we say that c has an *external circular tangent* if there is an open disc Δ_1

contained in the complement of X with $\overline{\Delta_1} \cap X = \{c\}$. It is easy to see from the geometry that this condition is equivalent to each of the following.

(i) For some $a \notin X$, $|c-a| = \min\{|z-a| : z \in X\}$.

(ii) There exists a real number θ such that for large numbers R we have $|1 + e^{i\theta}R(c-z)| > 1$ for all $z \in X \backslash \{c\}$.

Clearly every point of $[0,1]$ or $\bar{\Delta} \setminus \Delta$ has an external circular tangent.

Part B

We now consider a converse to Theorem 1, and again state the result for general perfect, compact subsets of **C**. The proof of this theorem is very similar to that of the corresponding theorem in [7].

Theorem 4: Let M_n be a nonanalytic weight and suppose $\phi \in D^{\infty}(X)$, $\phi : X \to X$ and $\limsup_{n\to\infty} (\frac{\|\phi^{(k)}\|}{k!})^{1/k}$ is finite. Suppose for some $b \in X$, $\phi(b)$ has an external circular tangent and $|\phi'(b)| > 1$. Also suppose that $D(X, M)$ is a Banach algebra. Then ϕ does not induce an endomorphism of $D(X, M)$.

Proof:(outline)

Assume that ϕ induces an endomorphism, $|\phi'(b)| > 1$ and that $\phi(b)$ has an external circular tangent. Let

$$F_R(z) = \frac{1}{1 + e^{i\theta}R(\phi(b) - z)}$$

where θ and R are chosen so $|1 + e^{i\theta}R(\phi(b) - z)| > 1$ for $z \in X \setminus \{\phi(b)\}$. Then $F_R \in D(X, M)$, $\|F_R\|_\infty = 1$ and $\|F_R^{(m)}\|_\infty = m!R^m$. With a very slight modification, namely replacing x_k by $\frac{\overline{\phi'(b)}}{|\phi'(b)|} \frac{\phi^{(k)}(b)}{k!}$ rather than by $\frac{\phi^{(k)}(b)}{k!}$, the proof proceeds exactly as in the proof of Theorem 3 of [7], eventually arriving at

$$\|F_R\|_{D(X,M)} = \sum_{m=0}^{\infty} \frac{m!R^m}{M_m}$$

and for each ϵ, $0 < \epsilon < 1$,

$$\|F_R \circ \phi\|_{D(X,M)} \geq \sum_{m=0}^{\infty} \frac{((1-\epsilon)R|\phi'(b)|)^m}{M_m}.$$

Since M_n is a nonanalytic weight, $\lim_{n\to\infty} (\frac{n!}{M_n})^{1/n} = 0$, and so $g(z) = \sum_{n=0}^{\infty} \frac{n!}{M_n} z^n$ is a transcendental entire function. In general, if $g(z) = \sum_{n=0}^{\infty} a_n z^n$

is a transcendental entire function and if $M_g(r) = \sup_{|z|=r} |g(z)|$, then for $c > 1$, $\lim_{r\to\infty} \frac{M_g(cr)}{M_g(r)} = \infty$. ([10], p 5, problem 24)

Also since $D(X, M)$ is complete, every endomorphism is bounded. Hence if ϕ induces an endomorphism, then there is a number $K > 0$ such that $\|f \circ \phi\|_{D(X,M)} \leq K\|f\|_{D(X,M)}$ for all $f \in D(X, M)$. Now for all $R > 0$,

$$M_g(R) = g(R) = \|F_R\|_{D(X,M)}$$

and

$$M_g((1-\epsilon)|\phi'(b)|R) = g((1-\epsilon)|\phi'(b)|R) \leq \|F_R \circ \phi\|_{D(X,M)}.$$

Thus, if ϕ induces an endomorphism, then for some $K > 0$,

$$M_g((1-\epsilon)|\phi'(b)|R) \leq K M_g(R)$$

for large R, and so

$$\limsup_{R\to\infty} \frac{M_g((1-\epsilon)|\phi'(b)|R)}{M_g(R)} \leq K < \infty.$$

This implies that $(1-\epsilon)|\phi'(b)| \leq 1$. Letting $\epsilon \to 0$ shows that $|\phi'(b)| \leq 1$ contrary to our assumption.

Remark: This theorem also has an interpretation when $D(X, M)$ is not complete. Suppose $\phi \in D^\infty(X)$, $\phi : X \to X$ and $\limsup_{n\to\infty}(\frac{\|\phi^{(k)}\|}{k!})^{1/k}$ is finite. Suppose for some $b \in X$, $\phi(b)$ has an external circular tangent and $|\phi'(b)| > 1$. Then ϕ does not induce a *bounded* endomorphism of $D(X, M)$.

For the next theorem, we call a compact plane set X *good* if $D(X, M)$ is complete for all nonanalytic algebra sequences M. Every compact set which is a finite union of uniformly regular sets in the sense of Dales and Davie is good.

Theorem 5: Let X be a good compact plane set, let $\phi : X \to X$ and let $\phi \in D^\infty(X)$. Suppose that there is a point $b \in X$ such that $|\phi'(b)| > 1$ and $\phi(b)$ has an external circular tangent. Then there is a nonanalytic sequence M_n with $\phi \in D(X, M)$ such that ϕ does not induce an endomorphism of $D(X, M)$.

Proof: Again we choose M_n inductively, but we also choose a sequence of complex numbers c_n in the complement of X. For any c off X we define F_c

in $D^\infty(X)$ by $F_c(z) = 1/(z-c)$. We now begin with $M_0 = M_1 = 1$ and any c_0, c_1 off X. For $n \geq 2$, and having chosen $M_0, \ldots, M_{n-1}$ and $c_0, \cdots, c_{n-1}$, we choose c_n and M_n as follows. Choose M_n such that (i) $M_n > \|\phi^{(n)}\|_\infty/2^n$, (ii) $M_n \geq (n!)^2$, (iii) $\frac{M_n}{M_k M_{n-k}} \geq \binom{n}{k}$ for $0 \leq k \leq n-1$, and such that for all $k < n$,

$$\|F_{c_k}^{(n)}\|_\infty/M_n < 2^{-n}\|F_{c_k}^{(k)}\|_\infty/M_k.$$

For the same $C_{n,\phi,m}$ as before, we have, for all $F \in D^\infty(X)$ and all $z \in X$,

$$|(F\circ\phi)^{(n)}(z)| \geq |F^{(n)}(\phi(z))||\phi'(z)|^n - \sum_{m=0}^{n-1} C_{n,\phi,m}\|F^{(m)}\|_\infty.$$

We now consider the functions F_c for suitable c. Since $\phi(b)$ has an external circular tangent, we can find c in the complement of X arbitrarily close to $\phi(b)$ and such that $|\phi(b) - c| = \min\{|z-c| : z \in X\}$. Choosing such $c = c_n$ close enough to $\phi(b)$ and considering the nature of the derivatives of F_c, we can arrange that

$$|(F_c\circ\phi)^{(n)}(\phi(b))| \geq \frac{1}{2}\|F_c^{(n)}\|_\infty|\phi'(b)|^n$$

and also that

$$\|F_c^{(n)}\|_\infty/M_n \geq \sum_{m=0}^{n-1}\|F_c^{(m)}\|_\infty/M_m.$$

The inductive choice may now proceed.

Having chosen the sequences M_n and c_n, we see that M_n is a nonanalytic sequence, and that $\phi \in D(X,M)$. Also, for $n \geq 2$, F_{c_n} is in $D(X,M)$, with $\|F_{c_n}\|_{D(X,M)} \leq 3\|F_{c_n}^{(n)}\|_\infty/M_n$. Further we have that $F_{c_n}\circ\phi \in D(X,M)$, so that

$$\|F_{c_n}\circ\phi\|_{D(X,M)} \geq |(F_{c_n}\circ\phi)^{(n)}(\phi(b))|/M_n \geq \frac{1}{2M_n}\|F_{c_n}^{(n)}\|_\infty|\phi'(b)|^n \geq \frac{|\phi'(b)|^n}{6}\|F_{c_n}\|_{D(X,M)}.$$

Since any endomorphism of $D(X,M)$ must be bounded, and $|\phi'(b)| > 1$ it follows that ϕ does not induce an endomorphism of $D(X,M)$.

Part C

We now look at the special case where $X = \bar{\Delta}$. We will use the property that elements in $D(\bar{\Delta},M)$ are analytic on Δ to obtain nearly complete results for the case when the self-maps are analytic on $\bar{\Delta}$.

Suppose ϕ is analytic on Δ and $\phi : \Delta \to \Delta$. It is well known that if ϕ_n denotes the n^{th} iterate of ϕ, then unless ϕ is a rotation, there is a unique point $z_0 \in \bar{\Delta}$ such that $\phi_n(z) \to z_0$ for all $z \in \Delta$. This point is known as the *Denjoy-Wolff point of* ϕ. If ϕ is continuous at z_0, and certainly if ϕ is analytic on $\bar{\Delta}$, then the Denjoy-Wolff point, z_0, is a fixed point of ϕ. If, in addition, $\phi'(z_0)$ exists, then $|\phi'(z_0)| \leq 1$. Further, for all other fixed points z' of ϕ, $\phi'(z') > 1$, whenever the derivative at z' exists. Thus for nonanalytic weights M_n, Theorem 4 shows that if $\phi \in D(\bar{\Delta}, M)$, ϕ is analytic on $\bar{\Delta}$, and $\phi : \bar{\Delta} \to \bar{\Delta}$ has two (or more) fixed points, then ϕ does not induce an endomorphism of $D(\bar{\Delta}, M)$.

We also recall that an inner function is an analytic function ϕ on Δ such that $|\phi(z)| \leq 1$ and $|\phi(e^{i\theta})| = 1$ for almost all θ.

The following classification of analytic functions $\phi : \bar{\Delta} \to \bar{\Delta}$ in terms of fixed points was shown in [5].

Proposition: Suppose $\phi : \bar{\Delta} \to \bar{\Delta}$ is analytic on $\bar{\Delta}$. Then the following mutually exclusive cases occur.

1. ϕ is inner.
2. ϕ is not inner and for all integers N there is no fixed point of ϕ_N on the unit circle.
3. There is a positive integer N for which $S_N = \{\phi_N(w) : |\phi_N(w)| = 1\}$ is finite, nonempty and every $z \in S_N$ is a fixed point of ϕ_N.
   (a) ϕ has no fixed point in Δ.
      i. $\phi'(z') < 1$ for some $z' \in S_N$ and $\phi_N'(z) > 1$ for $z' \neq z \in S_N$.
      ii. $\phi'(z') = 1$ for some $z' \in S_N$ and $\phi_N'(z) > 1$ for $z' \neq z \in S_N$.
   (b) ϕ_N (and hence ϕ) has a fixed point in Δ and $\phi_N'(z) > 1$ for all $z \in S_N$.

Theorem 6: Suppose M_n is a nonanalytic weight sequence and ϕ is an inner function in $D(\bar{\Delta}, M)$. Then ϕ does not induce an endomorphism of $D(\bar{\Delta}, M)$ unless ϕ is a constant or a rotation.

Proof:

Suppose that ϕ satisfies the hypothesis. Since ϕ is inner and is continuous on $\bar{\Delta}$, it follows that ϕ must be a finite Blaschke product. In particular, ϕ is analytic on $\bar{\Delta}$. We observe that if $\|\phi'\|_\infty > 1$, then there exists $b \in \bar{\Delta} \setminus \Delta$ such that $|\phi(b)| = 1$ and $|\phi'(b)| > 1$. Theorem 4 shows that ϕ does not

induce an endomorphism of $D(\bar{\Delta}, M)$ in this case. Thus we may assume that $\|\phi'\|_\infty \leq 1$. Let $N(\phi)$ denote the number of zeros of ϕ in Δ. Then

$$N(\phi) = \frac{1}{2\pi i}\int_{\bar{\Delta}\setminus\Delta} \frac{\phi'(z)}{\phi(z)}dz \leq \frac{1}{2\pi}\int_{\bar{\Delta}\setminus\Delta} \|\phi'\|_\infty |dz| \leq 1$$

since $|\phi(z)| = 1$ on the unit circle. Therefore $N(\phi) = 0$ or 1. If $N(\phi) = 0$, then ϕ is constant. Otherwise ϕ is a Möbius function, so $\phi(z) = e^{i\theta}\dfrac{z-\alpha}{1-\bar{\alpha}z}$ for some real θ and $\alpha \in \Delta$. Clearly,

$$\phi'(z) = e^{i\theta}\frac{1-|\alpha|^2}{(1-\bar{\alpha}z)^2}.$$

If $\alpha \neq 0$, then $\phi'(\dfrac{\alpha}{|\alpha|}) = \dfrac{1+|\alpha|}{1-|\alpha|} > 1$. Theorem 4 then implies that the map ϕ does not induce an endomorphism unless $\alpha = 0$ in which case ϕ has the form $\phi(z) = e^{i\theta}z$.

We remark that Theorem 6 also implies that the only automorphisms of $D(\bar{\Delta}, M)$ are those induced by rotations.

Another consequence is to show that the completeness of the algebra is needed in Theorem 4. For, if we let $D_r(\bar{\Delta}, M)$ denote the set of rational functions with poles off $\bar{\Delta}$, and M_n a nonanalytic weight, then the map T : $Tf(z) = f(\dfrac{2z-1}{z-2})$ induces an unbounded automorphism of the incomplete normed algebra $D_r(\bar{\Delta}, M)$. For, if T were bounded, then T would extend to a bounded automorphism of $D(\bar{\Delta}, M)$ induced by $\phi(z) = \dfrac{2z-1}{z-2}$, which is impossible by Theorem 6.

Theorem 7: Suppose M_n is a nonanalytic weight, ϕ is analytic on $\bar{\Delta}$, $\phi : \bar{\Delta} \to \bar{\Delta}$, and ϕ is not an inner function. Suppose that for all positive integers N there is no fixed point of ϕ_N on the unit circle. Then for some N_1, ϕ_N induces an endomorphism of $D(\bar{\Delta}, M)$ for $N \geq N_1$.

Proof:

It follows from the hypothesis that ϕ has a (unique) fixed point z_0 in Δ. Since $|\phi'(z_0)| < 1$, a compactness argument shows that $\phi_N(z) \in \Delta$ for all $z \in \bar{\Delta}$, N large. Theorem 2 implies that ϕ_N induces an endomorphism of $D(\bar{\Delta}, M)$.

However, we have the following example. Let $\phi(z) = \dfrac{1-z^3}{2}$. Here $\phi(-1) = 1$ and $\phi'(-1) = -\frac{3}{2}$. Therefore ϕ does not induce an endomorphism of $D(\bar{\Delta}, M)$, while $\phi_2(z) = \phi(\phi(z)) = \frac{7-3z^3+3z^6-z^9}{16}$ does induce an endomorphism since $\|\phi_2\|_\infty < 1$.

Theorem 8: Suppose ϕ is analytic on $\bar{\Delta}$, $\phi : \bar{\Delta} \to \bar{\Delta}$, $\phi \in D(\bar{\Delta}, M)$ and ϕ is not an inner function. If ϕ has a fixed point in Δ and ϕ_N has a fixed point on $\bar{\Delta} \setminus \Delta$ for some N, then ϕ does not induce an endomorphism of $D(\bar{\Delta}, M)$.

Proof:

Let N be a positive integer and z_1 be a fixed point of ϕ_N on $\bar{\Delta} \setminus \Delta$. Then $\phi_N'(z_1) > 1$ and so ϕ_N does not induce an endomorphism of $D(\bar{\Delta}, M)$. Clearly if ϕ_N does not induce an endomorphism, then ϕ does not induce an endomorphism.

In the case when all of the fixed points of ϕ lie on the boundary of $\bar{\Delta}$, we have the following.

Theorem 9: Suppose ϕ is analytic on $\bar{\Delta}$, $\phi \in D(\bar{\Delta}, M)$, $\phi : \bar{\Delta} \to \bar{\Delta}$ and ϕ has fixed points only on $\bar{\Delta} \setminus \Delta$.

(i) If ϕ or ϕ_N, for some N, has more than one fixed point on $\bar{\Delta} \setminus \Delta$, then ϕ does not induce an endomorphism of $D(\bar{\Delta}, M)$.

(ii) If ϕ has exactly one fixed point z_1 on $\bar{\Delta} \setminus \Delta$ and $\phi'(z_1) < 1$, then ϕ_N induces an endomorphism of $D(\bar{\Delta}, M)$ for all N large enough.

Proof:

(i) Let z_0 be the Denjoy-Wolff point of ϕ_N in $\bar{\Delta} \setminus \Delta$. Then $\phi_N'(z_0) \leq 1$. Then if z_1 is a second fixed point of ϕ_N, we have $|\phi_N(z_1)| = 1$ and $|\phi_N'(z_1)| > 1$. Theorem 4 then implies that ϕ_N and hence ϕ does not induce an endomorphism of $D(\bar{\Delta}, M)$.

(ii) Say ϕ has exactly one fixed point z_1 on $\bar{\Delta} \setminus \Delta$ and $\phi'(z_1) = r' < 1$. Let U be a neighborhood of z_1 for which $|\phi'(z)| < r = \dfrac{1 + r'}{2}$ and $\phi(z) \in U$ whenever $z \in U$. Let $A_n = \{z : \phi_n(z) \in U\}$. By the Denjoy-Wolff Theorem, $\bigcup_{n=0}^{\infty} A_n = \bar{\Delta}$. Since the set $\{A_n\}$ is nested, the compactness of $\bar{\Delta}$ shows that there exists K_0 so that $\phi_k(\bar{\Delta}) \subset U$ for $k > K_0$. Then for all $z \in \bar{\Delta}$, and integers $N > K_0$,

$$\phi_N'(z) = \phi'(\phi_{N-1}(z)) \cdots \phi'(\phi_{K_0}(z))\phi'(\phi_{K_0-1}(z)) \cdots \phi'(\phi(z))\phi'(z)$$

so that

$$\|\phi_N'\|_\infty < r^{N-K_0}(\|\phi'\|_\infty)^{K_0} = r^N(\frac{\|\phi'\|_\infty}{r})^{K_0}.$$

Then if $N > K_0$ is such that $r^N(\dfrac{\|\phi'\|_\infty}{r})^{K_0} < 1$, we have $\|\phi_N'\|_\infty < 1$ and so ϕ_N induces an endomorphism by Theorem 1.

Remark: There is still one case which is not resolved, namely, for $\phi \in D(\bar{\Delta}, M)$, $\phi : \bar{\Delta} \to \bar{\Delta}$ with $\phi(z_1) = z_1 \in \bar{\Delta} \setminus \Delta$, $\phi'(z_1) = 1$, $|\phi(z)| < 1$ for

$z \neq z_1$, but $\|\phi'\|_\infty > 1$. An example of such ϕ is the following.

$$\phi(z) = \frac{1}{2}[z + \frac{(1+i)z - 1}{z + (i-1)}].$$

However, Theorem 3 shows that for sufficiently rapidly growing M_n this map ϕ induces an endomorphism of $D(\bar{\Delta}, M)$.

We conclude with some open questions.

1. What is the complete answer for the $\phi's$ which do not satisfy the nice analytic properties that have been imposed?

2. The automorphisms of $D([0,1], M)$ are induced by $\phi(x) = x$ or $\phi(x) = 1-x$ and the automorphisms of $D(\bar{\Delta}, M)$ are induced by rotations. Can the automorphisms of other $D(X, M)$ be easily described? We remark that there are perfect, compact sets X such that the only automorphism of $D(X, M)$ is the identity operator.

3. What is the situation when X is a disconnected set such as the Cantor set?

4. The spectra of composition operators on Banach spaces of analytic functions on various domains have been studied in great detail. Can those methods be used to determine the spectra of endomorphisms of the algebras we have been considering?

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This research was supported by EPSRC grant GR/M31132